

A search for minimal 4d $\mathcal{N} = 1$ SCFT

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We discuss a candidate for a minimal interacting 4-dimensional $\mathcal{N} = 1$ superconformal field theory (SCFT). The model contains a chiral primary operator u satisfying the chiral ring relation $u^2 = 0$, and its scaling dimension is $\Delta(u) = 1.5$. The model is derived by turning on a $\mathcal{N} = 1$ preserving deformation of $\mathcal{N} = 2$ A_2 Argyres-Douglas theory. The central charges are given by $(a, c) = (263/768, 271/768) \simeq (0.342, 0.353)$. There is no moduli space of vacua, no flavor symmetry, and the chiral ring is finite.

I. INTRODUCTION

Recent progress of the conformal bootstrap program in dimensions larger than 2 (initiated in [1]) has provided many invaluable insights to the study of quantum field theory.

One of the interesting directions in those studies is to find a minimal interacting conformal field theory (CFT) in each dimension.¹ Candidates for such a minimal CFT might be Ising model in $(d = 3, \mathcal{N} = 0)$ [8], Wess-Zumino model in $(d = 3, \mathcal{N} = 2)$ [9], A_2 Argyres-Douglas theory in $(d = 4, \mathcal{N} = 2)$ [10], $SU(2)$ gauge theory in $(d = 4, \mathcal{N} = 4)$ [11], A_1 theory in $(d = 6, \mathcal{N} = (2, 0))$ [12], and so on.

In the cases listed above, it is not difficult to guess the minimal theory from the theories known in the literature. However, in the $(d = 4, \mathcal{N} = 1)$ case, even a possible candidate for such a minimal theory has not been so clear. In this letter, we initiate a search for such a theory.

Bootstrap searches of a minimal 4d SCFT [13–15] suggest that such a theory might have a chiral operator u satisfying the chiral ring relation $u^2 = 0$. The main purpose of this letter is to construct an explicit model having this chiral ring relation. The model also has rather small central charges a and c ; it is smaller than those of any known Lagrangian fixed points such as supersymmetric (SUSY) QCD. We hope that our work motivates a study of either (i) finding a smaller theory than the model presented in this letter, or (ii) making the bootstrap constraints sharper so as to confirm the minimality of the model.

II. FIELD THEORY CONSTRUCTION

One of the properties [13–15] which might be satisfied by a putative minimal SCFT is that it contains a

chiral primary operator u satisfying the chiral ring relation $u^2 = 0$. Our purpose is to construct a model satisfying that chiral ring relation. In such a model, the scaling dimension of u , which we denote as $\Delta(u)$, is currently bounded as $\Delta(u) > 1.41\dots$ [13–15]. The bound on $\Delta(u)$ is obtained by numerical analysis and may become sharper in the future analysis of conformal bootstrap. It turns out that $\Delta(u) = 1.5$ in our model. The model is obtained by a simple deformation of A_2 Argyres-Douglas (AD) theory [16].²

Let us first consider the $\mathcal{N} = 2$ AD theory [16, 18]. This theory contains an $\mathcal{N} = 2$ chiral supermultiplet, and in terms of $\mathcal{N} = 1$ supersymmetry, it contains two scalar chiral primaries and one spinor chiral primary as (u, λ_α, S) . Here u is the usual $\mathcal{N} = 2$ Coulomb moduli field.

$\mathcal{N} = 2$ SCFTs have R-symmetries $U(1)_R \times SU(2)_R$. Let us denote the Cartan generators of these R-symmetries as $R_{U(1)}$ and $R_{SU(2)}$, respectively. The R-charges of u are determined in [18] and the R-charges of λ_α and S follow from $\mathcal{N} = 2$ supersymmetry. They are listed in table I. In particular, the scaling dimension of u at $\mathcal{N} = 2$ superconformal fixed point is given by

$$\Delta_{\mathcal{N}=2}(u) = \frac{1}{2}R_{U(1)}(u) = \frac{6}{5}. \quad (1)$$

Now we consider $\mathcal{N} = 1$ deformation of this theory. We introduce a superpotential of the form

$$W \sim u^2. \quad (2)$$

From the above scaling dimension of u in the $\mathcal{N} = 2$ theory, we can see that this is a relevant deformation. We give some evidence that the theory flows to another non-trivial superconformal fixed point in the infrared (IR).

Assuming that there is no accidental $U(1)$ global symmetry in the IR, the $\mathcal{N} = 1$ R-charge is given by a linear combination as $xR_{U(1)} + (1-x)R_{SU(2)}$ for some parameter x . The superpotential (2) preserves only a particular

¹ In this letter we do not specify the precise measure of minimality of a theory. Bootstrap studies often constrain the central charge c defined by the two point function of the energy-momentum tensor. But it is natural to use a in 4d or F in 3d as a measure of the degrees of freedom, from the point of view of the a -theorem [2] and F -theorem [3, 4]. In fact, the c can increase along renormalization group flows in 4d [5, 6] and in 3d [7].

² Some deformation of $\mathcal{N} = 2$ AD theory was already considered in [16] (see also [17]). They considered the induced deformation on AD theory by turning on a $\mathcal{N} = 1$ deformation of UV $SU(3)$ gauge theory by $\text{tr } \Phi^3$. Their deformation is different from ours.

	u	λ_α	S
$R_{U(1)}$	$12/5$	$7/5$	$2/5$
$R_{SU(2)}$	0	1	2
R	1	$7/6$	$4/3$

TABLE I. R-charges of chiral operators.

combination of the R-symmetries given by

$$R = \frac{5}{12}R_{U(1)} + \frac{7}{12}R_{SU(2)}. \quad (3)$$

The R-charges of chiral operators under this R are also listed in table I. Assuming that the theory has a superconformal fixed point without accidental $U(1)$ symmetries, the scaling dimensions of chiral primary operators are related to the R-charges as

$$\Delta = \frac{3}{2}R. \quad (4)$$

One can see that all the chiral operators listed in table I satisfy the usual unitarity bounds $\Delta(u), \Delta(S) > 1$ (for scalar operators) and $\Delta(\lambda_\alpha) > 3/2$ (for spinor operators). This is one of the supports of our claim that the theory has a nontrivial fixed point. At the fixed point, we get the scaling dimension of u as

$$\Delta(u) = \frac{3}{2} = 1.5 \quad (5)$$

Let's now argue that our model is the minimal interacting SCFT among the theories which can be obtained from deformation of A_2 AD theory. The A_2 AD theory has three relevant scalar operators (u, S, u^2) with increasing scaling dimensions. The $\mathcal{N} = 1$ preserving deformation by u gives us a free chiral field in the IR [19]. Actually, one can check that the operator S hits the unitarity bound in this case. On the other hand, the deformation by S is the usual $\mathcal{N} = 2$ preserving deformation and one gets a free $\mathcal{N} = 2$ vector multiplet in the IR. Therefore, only the u^2 deformation gives us a nontrivial SCFT.

One might wonder if we could get more minimal theory by starting with other $\mathcal{N} = 2$ AD theories and turning on $\mathcal{N} = 1$ preserving relevant deformations of the form $W \sim u^2$ with u a Coulomb branch operator. An easy computation shows that our model has the minimal central charge among those possibilities. We will comment on the possibility of deformation $W \sim u$ in more general AD theories later.

A. Chiral ring relation

Now we show the chiral ring relation $u^2 = 0$ in the above model. What we will actually show is that u^2 is trivial as an element of the chiral ring,

$$u^2 = \bar{D}^2(\dots), \quad (6)$$

where the superspace notation follows that of Wess and Bagger. This equation holds once we introduce the superpotential (2), even without going to the IR fixed point. Then, at the IR fixed point, the usual argument in the superconformal field theory tells us that $u^2(x) := \lim_{y \rightarrow x} u(x) \times u(y) = 0$ in the operator product expansion (OPE) of $u \times u$.

To show (6), let us consider the following more general situation. Suppose that we have an $\mathcal{N} = 1$ SCFT \mathcal{S}_{UV} with a $U(1)$ flavor symmetry which we denote as $U(1)_F$. Let J be the current supermultiplet of this flavor symmetry. This operator satisfies the conservation equation

$$\bar{D}^2 J = 0. \quad (7)$$

Now we deform this theory by adding a superpotential

$$W = \eta \mathcal{O} \quad (8)$$

where \mathcal{O} is some chiral superfield and η is a parameter. If η is small, we can treat it as a perturbation. Then, in the deformed theory, the current conservation equation (7) is modified as

$$\bar{D}^2 J = \delta_F W = q_F \eta \mathcal{O}, \quad (9)$$

where δ_F means the transformation under $U(1)_F$, and q_F is the $U(1)_F$ charge of \mathcal{O} . This equation shows that \mathcal{O} is a trivial element of chiral ring of the theory once we turn on nonzero η if $q_F \neq 0$. Although this equation is derived by assuming that η is small, the qualitative feature that two protected operators J and \mathcal{O} combine to become an unprotected operator should be true for any finite η .

We can apply the above general argument to our theory. The UV theory \mathcal{S}_{UV} is the $\mathcal{N} = 2$ AD theory. We take the $U(1)_F$ charge F as $F = R_{U(1)} - R_{SU(2)}$ which is a flavor symmetry in terms of $\mathcal{N} = 1$ supersymmetry. We also take the operator \mathcal{O} as $\mathcal{O} = u^2$. This establishes the relation (6). Note that there is no flavor symmetry in the IR because $F = R_{U(1)} - R_{SU(2)}$ is broken.

Let us remark the following crucial point. Suppose that we have a chiral field Φ which is an elementary field in a Lagrangian field theory. If we introduce a superpotential $W \sim \Phi^2$, then not only Φ^2 but also Φ itself becomes trivial because we can simply integrate out the field Φ . Even if Φ is a composite field, it often happens that there exists Seiberg duality such that Φ becomes an elementary field in the dual side when the dimension $\Delta(\Phi)$ is rather small. In our construction, it is crucial that the AD theory is a kind of “non-Lagrangian” theory in the sense that u does not allow any interpretation as an elementary chiral field, but nevertheless u has a small scaling dimension ($\Delta^{\mathcal{N}=2}(u) = 6/5$) such that $W \sim u^2$ is a relevant deformation. Note that the u must be nontrivial because it can trigger an RG flow to a free theory [19].

B. Minimal models in 4d SCFT

We would like to point out an interesting feature of our model; it only has finitely many chiral primary operators.

First let us consider the chiral ring of the $\mathcal{N} = 2$ A_2 AD theory which is found at a point in the Coulomb branch of $SU(2)$ gauge theory with $N_f = 1$ flavor [18]. The chiral operators in the $SU(2)$ theory are given as

$$u \sim \text{tr } \Phi^2, \quad \lambda_\alpha \sim \text{tr}(\Phi W_\alpha), \quad S \sim \text{tr}(W^\alpha W_\alpha), \quad (10)$$

where Φ is the adjoint chiral field and W_α is the field strength chiral multiplet. At the classical level (see e.g., [20]), a chiral operator q in a representation ρ of a gauge group G satisfies $\rho(W_\alpha)q = 0$ as a chiral ring, and in particular we have $W_\alpha \Phi - \Phi W_\alpha = 0$ and $W_\alpha W_\beta + W_\beta W_\alpha = 0$. Furthermore, $\mathfrak{su}(2)$ matrices A, B, C, D satisfy $2 \text{tr}(ABCD) = \text{tr}(AB) \text{tr}(CD) - \text{tr}(AC) \text{tr}(BD) + \text{tr}(AD) \text{tr}(BC)$. Using these relations, we get classical ring relations

$$\lambda^\alpha \lambda_\alpha = uS, \quad \lambda_\alpha S = 0, \quad S^2 = 0. \quad (11)$$

In general, chiral ring relations can be modified quantum mechanically. However, it is reasonable to assume that there is a quantum chiral ring relation corresponding to each classical chiral ring relation. From the fact that quantum corrections must respect $U(1)_R \times SU(2)_R$ R-charges listed in Table I, we conjecture that (11) is valid quantum mechanically at the AD fixed point, up to corrections to the coefficient of $\lambda^\alpha \lambda_\alpha = uS$. One can also check that chiral operators involving the quarks of the $SU(2)$ $N_f = 1$ theory can be replaced by u, λ_α and S by using Konishi anomaly equations. For example, the $\mathcal{N} = 2$ current multiplet of $U(1)$ flavor symmetry is actually a descendant as pointed out in [18], and in $\mathcal{N} = 1$ language one has $\bar{Q}Q = S$, where Q and \bar{Q} are quarks.

After turning on the deformation (2), the operator u satisfies $u^2 = 0$ as shown above. Therefore, the only possible chiral operators of the model are u, λ_α, S and uS . Here we mainly use symmetry argument, and it is possible that some of these operators disappear in the chiral ring (but the u and S do not disappear because they can trigger nontrivial RG flows to free theories). It would be very desirable that one can verify our proposal using other methods.

A 4d SCFT with finitely many chiral primaries seems to be rare among the theories known in the literature. For example, such a theory cannot have any moduli space of vacua at all. In analogy with 2d minimal models, we might call 4d SCFTs as minimal models if the theory has only finitely many chiral primaries.

C. Central charges

The central charges of the model can be computed by using the results of the $\mathcal{N} = 2$ AD theory [21]. In $\mathcal{N} = 2$ SCFTs, the central charges $a_{\mathcal{N}=2}$ and $c_{\mathcal{N}=2}$ are related to the 't Hooft anomalies as

$$\text{tr } R_{U(1)}^3 = \text{tr } R_{U(1)} = 48(a_{\mathcal{N}=2} - c_{\mathcal{N}=2}), \quad (12)$$

$$\text{tr } R_{U(1)} R_{SU(2)}^2 = 8(2a_{\mathcal{N}=2} - c_{\mathcal{N}=2}) \quad (13)$$

On the other hand, in $\mathcal{N} = 1$ theories, a and c are given in terms of 't Hooft anomalies of the R-symmetry as [6]

$$a = \frac{1}{32}(9 \text{tr } R^3 - 3 \text{tr } R), \quad c = \frac{1}{32}(9 \text{tr } R^3 - 5 \text{tr } R). \quad (14)$$

Therefore, if we know $a_{\mathcal{N}=2}$ and $c_{\mathcal{N}=2}$, we can easily compute the central charges of the IR fixed point by using the relation (3). Such a calculation was done in [19] in the case of the superpotential deformation $W \sim u$, and it was confirmed that the central charges are those of a free chiral multiplet in that case.

The central charges in the $\mathcal{N} = 2$ AD theory are given by [21]

$$a_{\mathcal{N}=2} = \frac{43}{120} \simeq 0.358, \quad c_{\mathcal{N}=2} = \frac{11}{30} \simeq 0.367. \quad (15)$$

By using this and (3), we get the a and c of our model,

$$a = \frac{263}{768} \simeq 0.342, \quad c = \frac{271}{768} \simeq 0.353. \quad (16)$$

Note that the a satisfies the a -theorem $a < a_{\mathcal{N}=2}$ [2]. Note also that the central charges satisfy the bound $1/2 < a/c < 3/2$ [22]. These facts support our claim that the superpotential deformation (2) leads to a nontrivial SCFT in the IR limit without accidental symmetries.

The value of c obtained in (16) is somewhat larger than the current lower bound $c \gtrsim 0.1$ obtained in [15]. A possible reason is that our model has more than one chiral operator while the bootstrap method uses only one chiral operator. However, the central charges of our model are smaller than those of conventional SUSY QCD fixed points. For example, the central charges of the $SU(2)$ theory with $N_f = 4$ flavors is given by $(a, c) = (3/4, 17/16) \simeq (0.75, 1.06)$.

III. DISCUSSION

We have discussed an interesting model which has many desirable properties to be a minimal model. However, there is still a gap between our model and the bootstrap constraints and it is very interesting to close the gap from bootstrap side and/or model search side.

There are at least two ways to search for 4d minimal models. Let us describe preliminary results, leaving more detailed study for a future work. The first method is to engineer four dimensional $\mathcal{N} = 1$ theory by putting 6d $\mathcal{N} = (2, 0)$ theory on a punctured Riemann surface C with line bundles (L_1, L_2) such that $L_1 \otimes L_2 = T^*C$. [23–25] (see also [26, 27]). To get $\mathcal{N} = 2$ AD theories [28], one chooses a sphere with one irregular puncture or a sphere with one irregular and one regular puncture, and the choice of bundle is $(L_1, L_2) = (\mathcal{O}(-2), \mathcal{O})$. To get $\mathcal{N} = 1$ AD theory, one has more choices of bundles and punctures.

Let us consider the case of 6d A_1 theory with $(L_1, L_2) = (\mathcal{O}(-1), \mathcal{O}(-1))$. We use an irregular puncture of the following form

$$\Phi_1 \sim \frac{1}{z^{1+1/2}} \text{diag}(1, -1), \quad \Phi_2 \sim \frac{1}{z^{1+1/2}} \text{diag}(1, -1). \quad (17)$$

The spectral curve of this system can be easily computed [29]. By putting the puncture at $z = \infty$, we get

$$v^2 = z + u_1, \quad vw = z + u_2, \quad w^2 = z + u_3. \quad (18)$$

We may use the automorphisms of the Riemann sphere to set $u_3 = 0$. We also need to impose the relation $(z + u_2)^2 = (z + u_1)z$, and this gives the chiral ring relation

$$u_1 = 2u_2, \quad u_2^2 = 0. \quad (19)$$

So there is a chiral operator u_2 satisfying the condition $u_2^2 = 0$. To find the scaling dimension of the operator u_2 , we impose the condition that the canonical three form $\Omega = dv \wedge dw \wedge dz$ has scaling dimension 3. We find $\Delta(v) = \Delta(w) = 3/4$, $\Delta(z) = 3/2$, and $\Delta(u_2) = 3/2$. These features are the same as the field theory model, and it will be very interesting to determine whether these models are the same or not.

Another direction is to study deformations of more general AD theories by a linear deformation $W \sim u$. In this case it is not clear whether there is no accidental symmetry in the IR (see [19] for some examples which flow to free theories). Assuming the absence of accidental symmetry, we find a model with smaller central charges. We start with A_4 AD theory, and turn on $W \sim u$ deformation with $\Delta_{N=2}(u) = \frac{10}{7}$. We get $(a, c) = (633/2000, 683/2000) \simeq (0.316, 0.341)$ [27]. It is very interesting to determine whether this theory really flows to a nontrivial SCFT without accidental symmetry, and if so, determine its chiral ring.

Note added: After the first version of this paper was submitted, the paper [30] appeared which has overlap. There, the relation $u\lambda_\alpha = 0$ was also discussed.

IV. SUMMARY

Searching for minimal nontrivial quantum field theories is extremely interesting. In many cases the theories are intrinsically strongly coupled, and conventional textbook approaches do not work at all. In two-dimensions, a beautiful story about minimal models has been developed starting from [31] by conformal bootstrap methods, and the studies in higher dimensions have begun recently.

In this letter we have initiated the search of minimal models in four-dimensional $\mathcal{N} = 1$ supersymmetric theories by explicitly constructing a candidate. We gave a possible definition of minimal models in terms of the finiteness of a certain class of operators (i.e., chiral primaries), and discussed some evidence that our theory satisfy that minimality condition. We have not given a complete proof of the minimality, but a proof might be possible by using techniques such as in [30]. There is still a gap between our model and the bounds obtained by numerical conformal bootstrap [13–15]. The existence of such a gap suggests that there is room to learn something new about the nature of strongly coupled field theories in general. Therefore, it is very important to fill this gap in the future.

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